17

Methods for Vibration Analysis

§17.1 PROBLEM CLASSIFICATION

According to S. H. Krandall (1956), engineering problems can be classified into three categories:

- equilibrium problems
- eigenvalue problems
- propagation problems

Equilibrium problems are characterized by the structural or mechanical deformations due to quasi-static or repetitive loadings. In other words, in structural and mechanical systems the solution of equilibrium problems is a stress or deformation state under a given load. The modeling and analysis tasks are thus to obtain the system stiffness or flexibility so that the stresses or displacements *computed* accurately match the observed ones.

Eigenvalue problems can be considered as extentions of equilibrium problems in that their solutions are dictated by the same equilibrium states. There is an additional distinct feature in eigenvalue problems: their solutions are *characterized* by a unique set of system configurations such as resonance and buckling.

Propagation problems are to predict the subsequent stresses or deformation states of a system under the time-varying loading and deformation states. It is called *initial-value problems* in mathematics or disturbance transmissions in wave propagation.

Modal testing is perhaps the most widely accepted words for activities involving the characterization of mechanical and structural vibrations through testing and measurements. It is primarily concerned with the determination of mode shapes (eigenvevtors) and modes (eigenvalues), and to the extent possible the damping ratios of a vibrating system. Therefore, modal testing can be viewed as experimental solutions of eigenvalue problems.

There is one important distinction between eigenvalue analysis and modal testing. Eigenvalue analysis is to obtain the eignvalues and eigenvectors from the *analytically constructed governing equations* or from a given set of mass and stiffness properties. There is no disturbance or excitation in the problem description. On the other hand, modal testing is to seek after the same eigenvalues and eigenvectors by injecting disturbances into the system and by measuring the system response. However, modal testing in earlier days tried to measure the so-called *free-decay responses* to mimick the steady-state responses of equilibrium problems.

Table 1: Comparison of Engineering Analysis and System Identification

	Engineering Analysis	System Identification
Equilibrium	Construct the model first, then obtain deformations under any given load.	Measure the dynamic input/output first, then obtain the flexibility.
Eigenvalue	Construct the model first, then obtain eigenvalues without any specified load.	Measure the dynamic input/output first, then obtain eigenvalues that corresponds to the specific excitation.
Propagation	Construct the model first, then obtain responses for time-varying loads.	Measure the dynamic input/output first, then obtain the model corresponds to the specific load

Observe from the above Table that the models are first constructed in engineering analysis. In system identification the models are constructed only after the appropriate input and output are measured. Nevertheless, for both engineering analysis and system identification, *modeling* is a central activity. Observe also that, in engineering analysis, once the model is constructed it can be used for all of the three problems. On the other hand, the models obtained by system identification are usually valid only under the specific set of input and output pairs. The extent to which a model obtained through system identification can be applicable to dynamic loading and transient response measurements depends greatly upon the input characteristics and the measurement setup and accuracy.

§17.2 STRUCTURAL MODELING BY SYSTEM IDENTIFICATION

As noted in the previous section, modeling constitutes a key activity in engineering analysis. For example, the finite element method is a discrete structural modeling methodology. Structural system identification is thus a complementary endeavor to discrete modeling techniques. A comprehensive modeling of structural systems is shown in Fig. 1. The entire process of structural modeling is thus made of seven blocks and seven information transmission arrows (except the feedback loop).

Testing consists of the first two blocks, *Structures* and *Signal Conditioning* along with three actions, the application of disturbances as input to the structures, the collection of sensor output, and the processing of the sensor output via filtering for noise and aliasing treatment.

FFT and **Wavelets Transforms** are software interface with the signal conditioniners. From the viewpoint of system identification, its primary role is to produce as accurately as possible *impulse response functions* either in frequency domain or in time domain variables. It is perhaps the most important software task because all the subsequent system realizations and the determination of structural model parameters *do* depend on the extracted impulse response data. About a fourth of this course will be devoted to learn methods and techniques for extracting the impulse response functions.

System realization performs the following task:

For the model problem of

plant: $\dot{\mathbf{x}} = \mathcal{A}\,\mathbf{x} + \mathcal{B}\,\mathbf{u}$

Given measurements of

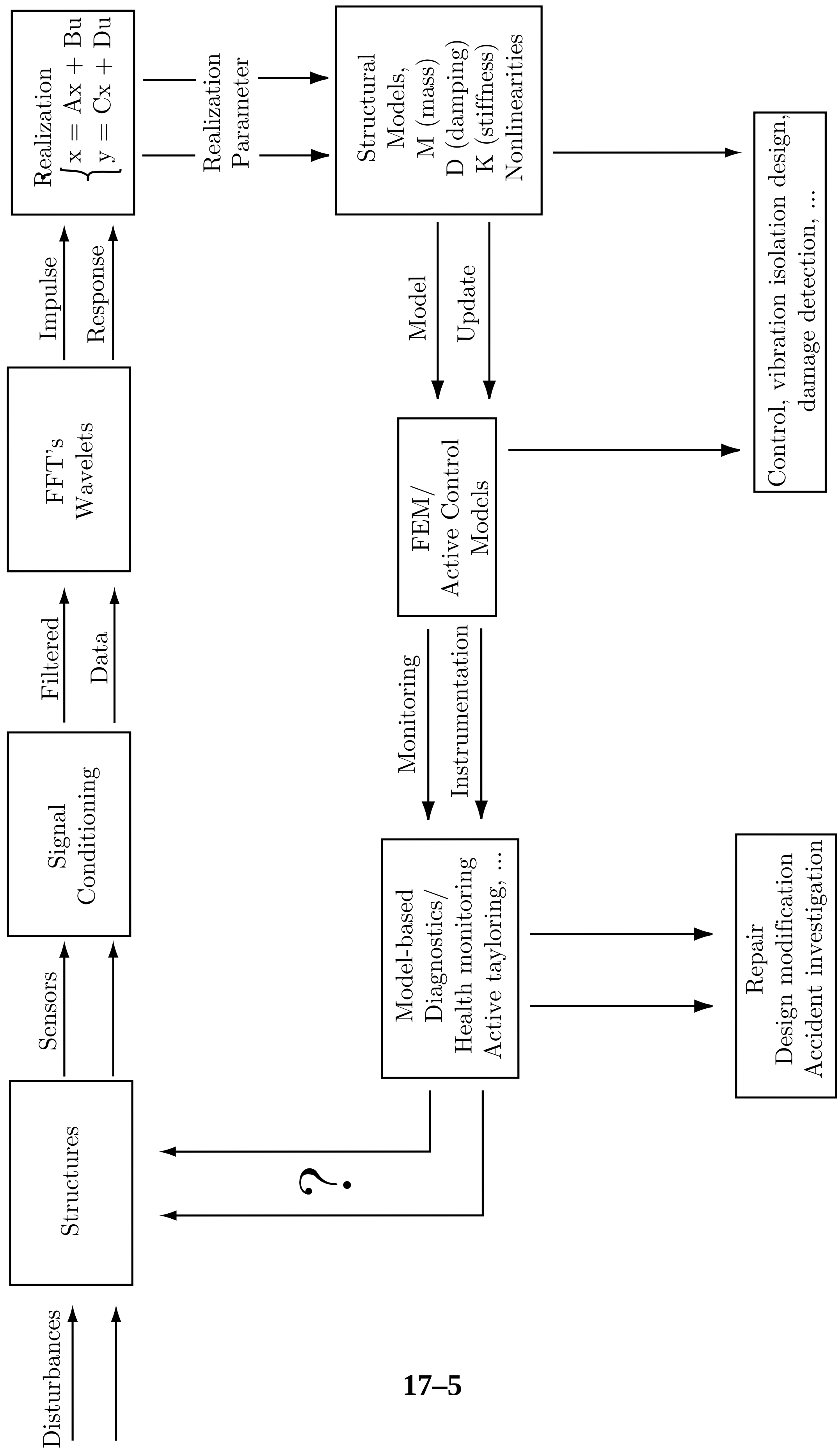
output: $\mathbf{y} = \mathcal{C}\,\mathbf{x} + \mathcal{D}\,\mathbf{u}$

input: $\mathbf{u}$

Determine

system characteristics: $\mathcal{A}$, $\mathcal{B}$ $\mathcal{C}$ and $\mathcal{D}$

Structural modeling block is to extract physical structural quantities from the system characteristics or realization parameters ($\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$). This is because


Disturbances
Structures
Sensors
Signal Conditioning
Filtered Data
FFT's Wavelets
Impulse Response
Realization
$\begin{cases} x = Ax + Bu \\ y = Cx + Du \end{cases}$
Realization Parameter
Structural Models,
M (mass)
D (damping)
K (stiffness)
Nonlinearities
Model Update
FEM/ Active Control Models
Monitoring Instrumentation
Model-based Diagnostics/
Health monitoring
Active tayloring, ...
?
Control, vibration isolation design, damage detection, ...
Repair
Design modification
Accident investigation

realization characteristics still consist of abstract mathematical models, not necessarily in terms of the desired structural quantities. Specifically, one obtains

Given

realization parameters: $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$

Determine either

modal quantities: modes($\boldsymbol{\omega}$) and mode shapes (ϕ)

or physical matrices: mass ($\mathbf{M}$), stiffness($\mathbf{K}$) and damping($\mathbf{D}$)

Finite element model updating, active controls and health monitoring are the beneficiaries of the preceding four activities. Hence, we will try to touch upon these topics, perhaps as term projects, depending on how this course progresses itself before the Thanksgiving recess.

Finally, if necessary, one may have to repeat testing, hopefully this time utilizing the experience gained from the first set of activities. Even experienced experimentalists often must repeat testing. A good experimentalist rarely believes his/her initial results whereas a typical analyst almost always thinks his/her initial results are valid!

§17.3 ANALYTICAL SOLUTION OF VIBRATING STRUCTURES

This section is a starting point of a guided tour, though an incomplete one at best, of modeling, analysis and structural system identification. To this end, we introduce a reference problem, which for our case is an analytically known model so that when we are astray from the tour path, we can all look up the map and hopefully steer ourselves back to the reference point and continue our tour.

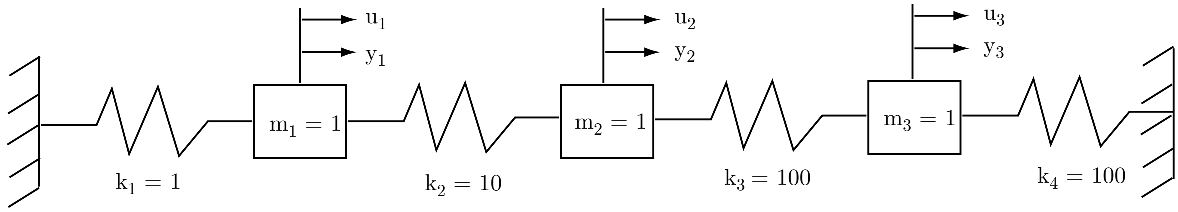


Figure 2. Three DOF Spring-Mass System

§17.3.1 Three Degrees of Freedom Model Problem

The model problem we are going to walk through is a 3-DOF (three degrees of freedom), undamped oscillator given by

The system mass and stiffness matrices, **M** and **K** are given by

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$\mathbf{K} = \begin{bmatrix} 11 & -10 & 0 \\ -10 & 110 & -100 \\ 0 & -100 & 200 \end{bmatrix} \tag{17.1}$$

whose frequencies (the square root of the eigenvalues) are given by

$$\mathbf{\Omega} = [\, 2.991168982 \quad 6.875901898 \quad 16.271904658 \,] \qquad \text{rad/sec.} \tag{17.2}$$

and the eigenvectors are given by

$$\phi = \begin{bmatrix} 0.974189634 & 0.224712761 & 0.021417986 \\ 0.199992181 & -0.815213561 & -0.543534706 \\ 0.104678951 & -0.533789306 & 0.839113397 \end{bmatrix} \tag{17.3}$$

The mode shapes (eigenvectors) are plotted below in Fig. 3. Let us now compute the analytical impulse response functions.

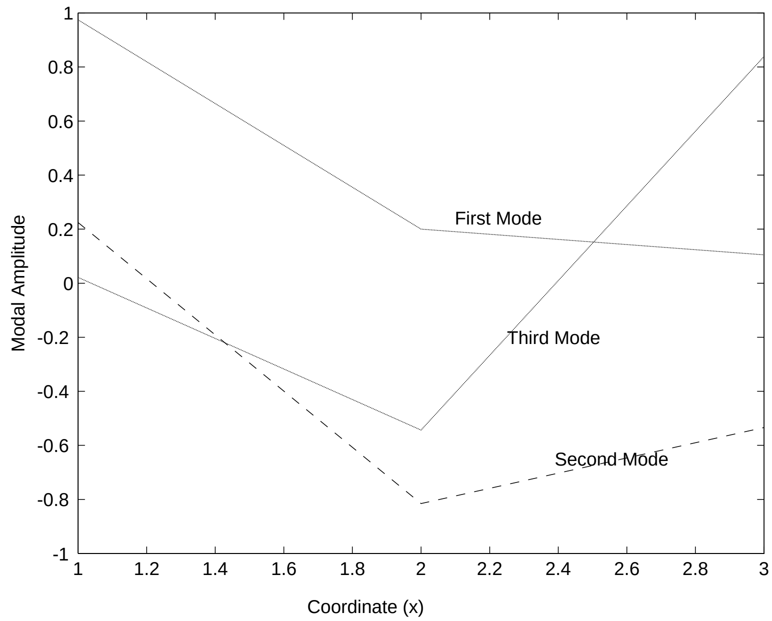


Figure 3. Mode Shapes of Three DOF Spring-Mass System

§17.3.2 Impulse Response Functions

For a given forcing function $\mathbf{f}(t)$, the model equations are given by

$$\mathbf{M}\,\ddot{\mathbf{q}} + \mathbf{K}\,\mathbf{q} = \mathbf{f}(t) \tag{17.4}$$

where $\mathbf{q}$ is the displacement vector of its dimension (3×1). Let us consider the following *special matrix forcing* function

$$\mathbf{f} = \begin{bmatrix} \delta(t) & 0 & 0 \\ 0 & \delta(t) & 0 \\ 0 & 0 & \delta(t) \end{bmatrix} \tag{17.5}$$

where the unit impulse function $\delta(t)$ is defined by

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1 \tag{17.6}$$

Notice we have introduced a *matrix-valued* forcing function instead of the customary *vector-valued* forcing function. Hence, the response or output $\mathbf{q}$ should be a matrix-valued function. The forcing fucntion or input consists of three separate unit impulse loadings, each applied at one of the three distinct mass locations. This is in fact the most desired testing condition called *single input multiple output* (SIMO) testing procedure.

There are two ways of characterizing the impulse response functions: frequency domain and time doamin characterizations. As we are planning to study both characterizations, we will describe them for the example 3-DOF problem.

§1.3.2.1 Frequency Response Functions

In order to obtain the frequency response functions of the model 3-DOF problem (17.3), we first seek a solution of the form

$$\mathbf{q} = \mathbf{q}_0 e^{j\omega t} \tag{17.7}$$

Upon substituting into (1.4) one obtains

$$\mathbf{q}(t) = (-\boldsymbol{\omega}^2\mathbf{M} \ + \mathbf{K})^{-1}\mathbf{f}(t) \tag{17.8}$$

Fourier transformations of both sides of the above expression yield

$$\mathbf{Q}(\omega) = \int_{-\infty}^{\infty} \mathbf{q}(t)e^{-j\omega t}\, dt = \int_{-\infty}^{\infty} (-\boldsymbol{\omega}^2\mathbf{M} \ + \mathbf{K})^{-1}\mathbf{f}(t)e^{-j\omega t}\, dt \tag{17.9}$$

Since a convolution of any function with the impulse response function $\delta(t)$ is the function itself, we have

$$\mathbf{Q}(\omega) = \mathbf{H}(\omega) = (-\boldsymbol{\omega}^2\mathbf{M} \ + \mathbf{K})^{-1} \tag{17.10}$$

There are a total of nine components in the impulse response function $\mathbf{H}(\omega)$ for this example problem. This can be seen by expanding $\mathbf{H}(\omega)$

$$\mathbf{H}(\omega) = \begin{bmatrix} \mathbf{H}_{11} & \mathbf{H}_{12} & \mathbf{H}_{13} \\ \mathbf{H}_{21} & \mathbf{H}_{22} & \mathbf{H}_{23} \\ \mathbf{H}_{31} & \mathbf{H}_{32} & \mathbf{H}_{33} \end{bmatrix} \tag{17.11}$$

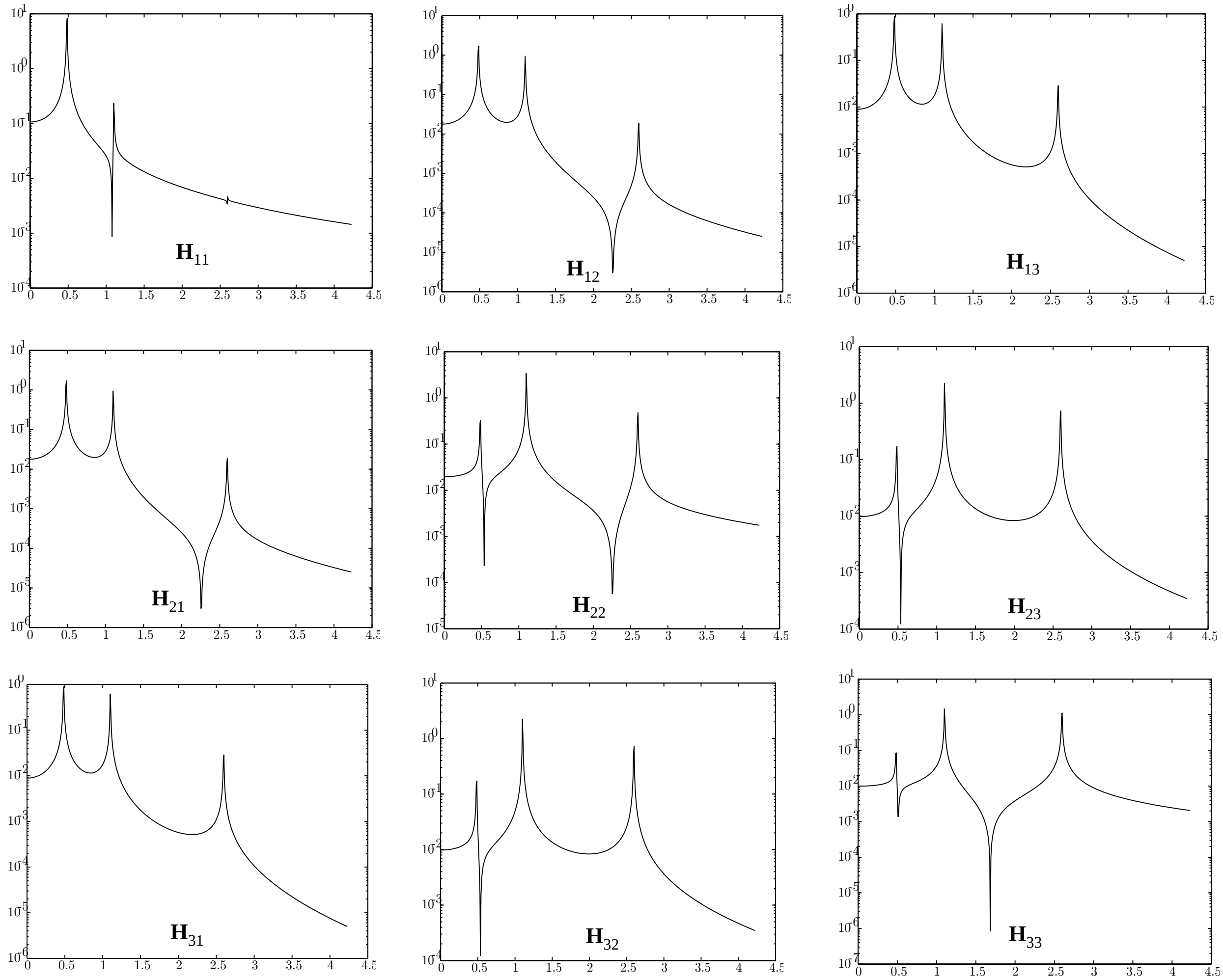


Figure 4. Frequency Response Functions, $\mathbf{H}(\omega)$

For example, $\mathbf{H}_{11}$ is the frequency response function at the mass point 1 where the unit impulse is applied. On the other hand, $\mathbf{H}_{12}$ and $\mathbf{H}_{13}$ are due to the unit impulse load at mass points 2 and 3, respectively. In general, $\mathbf{H}_{rs}$ is the frequency response of the rth degrees freedom due to the load applied at the mass point s. Figure 4 shows all of the nine components vs. frequency ($f = \omega/2\pi$).

Notice that we have the symmetry of the following three components:

$$\begin{aligned} \mathbf{H}_{12} &= \mathbf{H}_{21} \\ \mathbf{H}_{23} &= \mathbf{H}_{32} \\ \mathbf{H}_{13} &= \mathbf{H}_{31} \end{aligned} \tag{17.12}$$

which can be verified from Fig. 4. This is due to the fact that the unit impulse loads applied at the three mass points are the same, and both the mass and stiffness matrices are symmetric. This property plays an important role in determining the frequency response functions from measured data.

§1.3.2.2 Time-Domain Impulse Response Functions

Theoretically, the impulse response function $\mathbf{h}(t)$ has to be determined from the following convolution integral

$$\mathbf{q} = \int_{-\infty}^{t} \mathbf{h}(t - \tau)\, \mathbf{f}(\tau)\, d\tau \tag{17.13}$$

Luckily for the impulse load given by (17.5), one can obtain the time-domain impuse response function $\mathbf{h}(t)$ either by an analytical approach such as the transition method and modal superposition technique using the system eigenvalues and their eigenvectors, or by a direct time integration method. If one chooses the analytical transition matrix technique, one first transforms the second-order systems into a first-order equation:

$$\begin{aligned}
\dot{\mathbf{x}} &= \mathcal{A}\,\mathbf{x} + \mathcal{B}\,\mathbf{u} \\
\mathcal{A} &= \begin{bmatrix} \mathbf{0} & -\mathbf{K} \\ \mathbf{M}^{-1} & \mathbf{0} \end{bmatrix} \\
\mathcal{B} &= \begin{bmatrix} \mathbf{I} \\ \mathbf{0} \end{bmatrix} \\
\mathbf{u} &= \mathbf{f} \\
\mathbf{x} &= [\,\mathbf{v} \quad \mathbf{q}\,]^T, \qquad \mathbf{v} = \mathbf{M}\dot{\mathbf{q}}
\end{aligned} \tag{17.14}$$

Solution of the above equation is given by

$$\mathbf{x} = e^{\mathcal{A}(t - t_0)}\mathbf{x}(0) + \int_{t_0}^{t} e^{\mathcal{A}(t - \tau)}\, \mathcal{B}\,\mathbf{u}(\tau)\, d\,\tau \tag{17.15}$$

Alternatively, one may invoke the classical modal superposition method. To this end, we introduce

$$\mathbf{q} = \phi\, \boldsymbol{\eta} \tag{17.16}$$

Substituting this into the coupled dynamic equation (17.4) gives

$$\boldsymbol{\phi}^T\ \mathbf{M}\boldsymbol{\phi}\ \ddot{\boldsymbol{\eta}} + \boldsymbol{\phi}^T\ \mathbf{K}\boldsymbol{\phi}\ \boldsymbol{\eta} = \boldsymbol{\phi}^T\ \mathbf{f} \tag{17.17}$$

Using the identities

$$\boldsymbol{\phi}^T\ \mathbf{M}\boldsymbol{\phi}\ = \mathbf{I}, \quad \boldsymbol{\phi}^T\ \mathbf{K}\boldsymbol{\phi}\ = \text{diag}\ (\boldsymbol{\Omega}^2) \tag{17.18}$$

the modal equation can be expressed as

$$\ddot{\boldsymbol{\eta}} + \boldsymbol{\Omega}^2\boldsymbol{\eta} =\ \mathbf{0} \tag{17.19}$$

with the following modal initial conditions:

$$\boldsymbol{\eta}(0) = \mathbf{0}, \qquad \dot{\boldsymbol{\eta}}(0) =\ \boldsymbol{\phi}^T\mathbf{M}^{-1} \tag{17.20}$$

Note that the impulse loads $\boldsymbol{\phi}^T\mathbf{f}$ is replaced by the equivalent initial velocity condition by the momentum conservation relation

$$\begin{aligned} &\dot{\mathbf{q}}(0) = \dot{\mathbf{q}}(0_-) + \triangle t\ \ddot{\mathbf{q}} = \mathbf{M}^{-1}\int_0^{\triangle t} \mathbf{f}\ d\ t, \qquad \dot{\mathbf{q}}(0_-) = 0 \\ &\dot{\boldsymbol{\eta}}(0) = \boldsymbol{\phi}^{-1}\dot{\mathbf{q}}(0) = \boldsymbol{\phi}^T\dot{\mathbf{q}}(0) = \boldsymbol{\phi}^T\mathbf{M}^{-1} \end{aligned} \tag{17.21}$$

Solution of (17.19) with the initial condition (17.20) is given by

$$\mathbf{q}(t) = \mathbf{h}(t) = \boldsymbol{\phi} \begin{bmatrix} \sin\Omega_1 t/\Omega_1 & 0 & 0 \\ 0 & \sin\Omega_2 t/\Omega_2 & 0 \\ 0 & 0 & \sin\Omega_3 t/\Omega_3 \end{bmatrix} \boldsymbol{\phi}^T \tag{17.22}$$

Finally, a direct time integration, e.g., by using the trapezoidal rule can give an approximate solution in the form

Given the initial condition and:

$$\delta = dt/2, \qquad \delta 2 = \delta * \delta$$
$$\mathbf{S} = \mathbf{M} + \delta 2 * \mathbf{K}$$

Start integration:

$$\begin{aligned} t &= t + 2\delta \\ \mathbf{g} &= \mathbf{M}(\mathbf{q}^n + \delta \dot{\mathbf{q}}^n) \\ \mathbf{q}^{n+\frac{1}{2}} &= \mathbf{S}^{-1}\mathbf{g} \\ \mathbf{q}^{n+1} &= 2\mathbf{q}^{n+\frac{1}{2}} - \mathbf{q}^n \\ \ddot{\mathbf{q}}^{n+1} &= -\mathbf{M}^{-1}\mathbf{K}\mathbf{q}^{n+1} \\ \dot{\mathbf{q}}^{n+1} &= \dot{\mathbf{q}}^n + \delta(\ddot{\mathbf{q}}^{n+1} + \ddot{\mathbf{q}}^n) \end{aligned} \tag{17.23}$$

End the integration

The time-domain impulse response functions are given by $\mathbf{q}$:

$$\mathbf{h}(t) = \mathbf{q}(t) = \begin{bmatrix} \mathbf{h}_{11} & \mathbf{h}_{12} & \mathbf{h}_{13} \\ \mathbf{h}_{21} & \mathbf{h}_{22} & \mathbf{h}_{23} \\ \mathbf{h}_{31} & \mathbf{h}_{32} & \mathbf{h}_{33} \end{bmatrix} \tag{17.24}$$

Notice also that we have again the following symmetry from Fig. 5 that corresponds to the result obtained by the frequency method (17.12):

$$\begin{aligned} \mathbf{h}_{12} &= \mathbf{h}_{21} \\ \mathbf{h}_{23} &= \mathbf{h}_{32} \\ \mathbf{h}_{13} &= \mathbf{h}_{31} \end{aligned} \tag{17.25}$$

§17.4 DISCRETE IMPULSE RESPONSE FUNCTIONS

In the preceding section we have obtained the *reference* impulse response functions both in frequency and time domains when the Fourier transforms can be carried

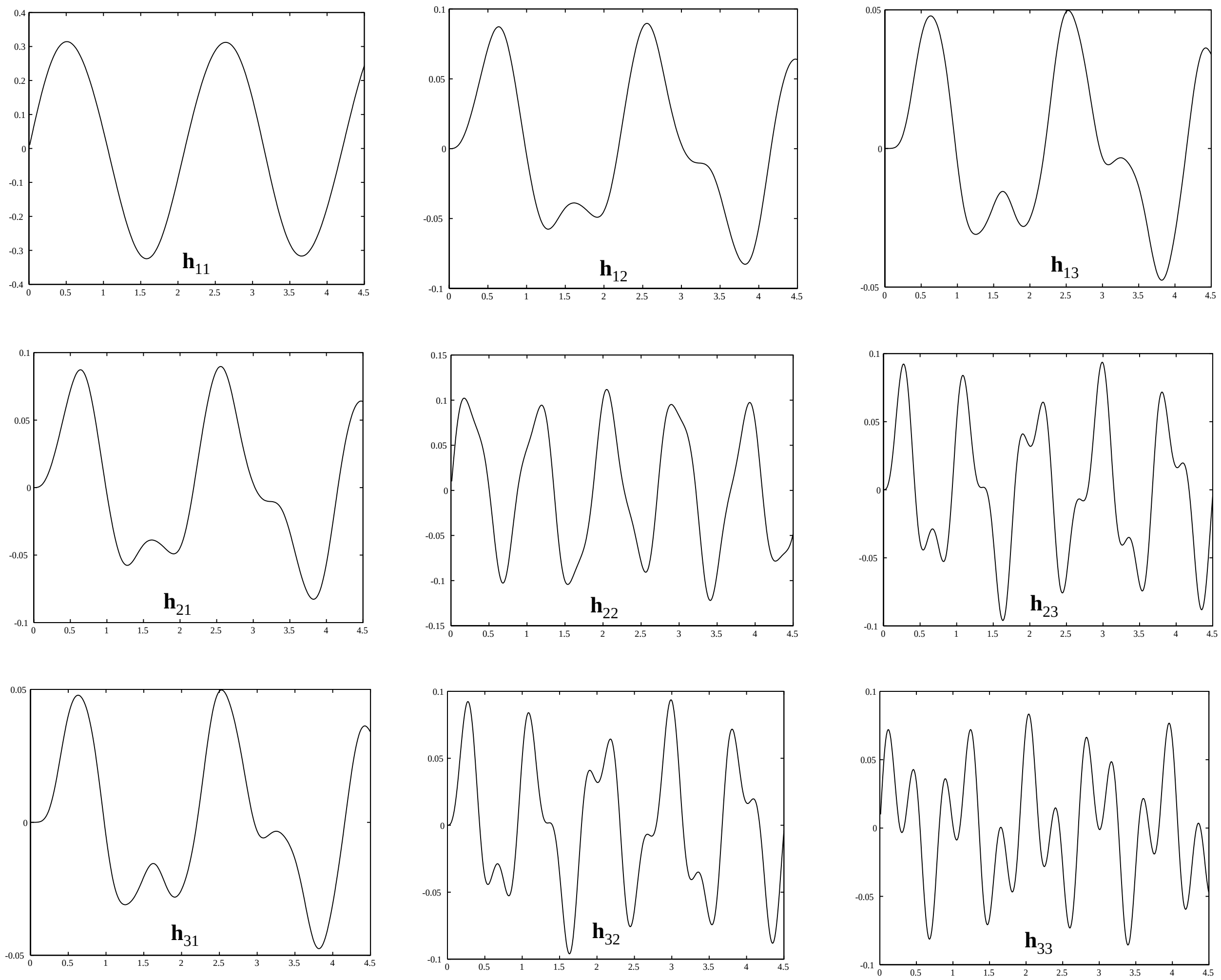


Figure 5. Time-Domain Impulse Rsponse Functions , $\mathbf{h}(t)$

out exactly. In practice both the forward and inverse Fourier transforms are caried out by Fast Fourier Transform (FFT) procedure.

In order to appreciate the impact of discrete approximations on the solution accuracy, let us consider the first solution component of the modal response equation (17.19) given by

$$\eta(t)_1 = \sin(2\pi f_1 t)/(2\pi f_1), \qquad f_1 = \text{ 4.760593E-01} \tag{17.26}$$

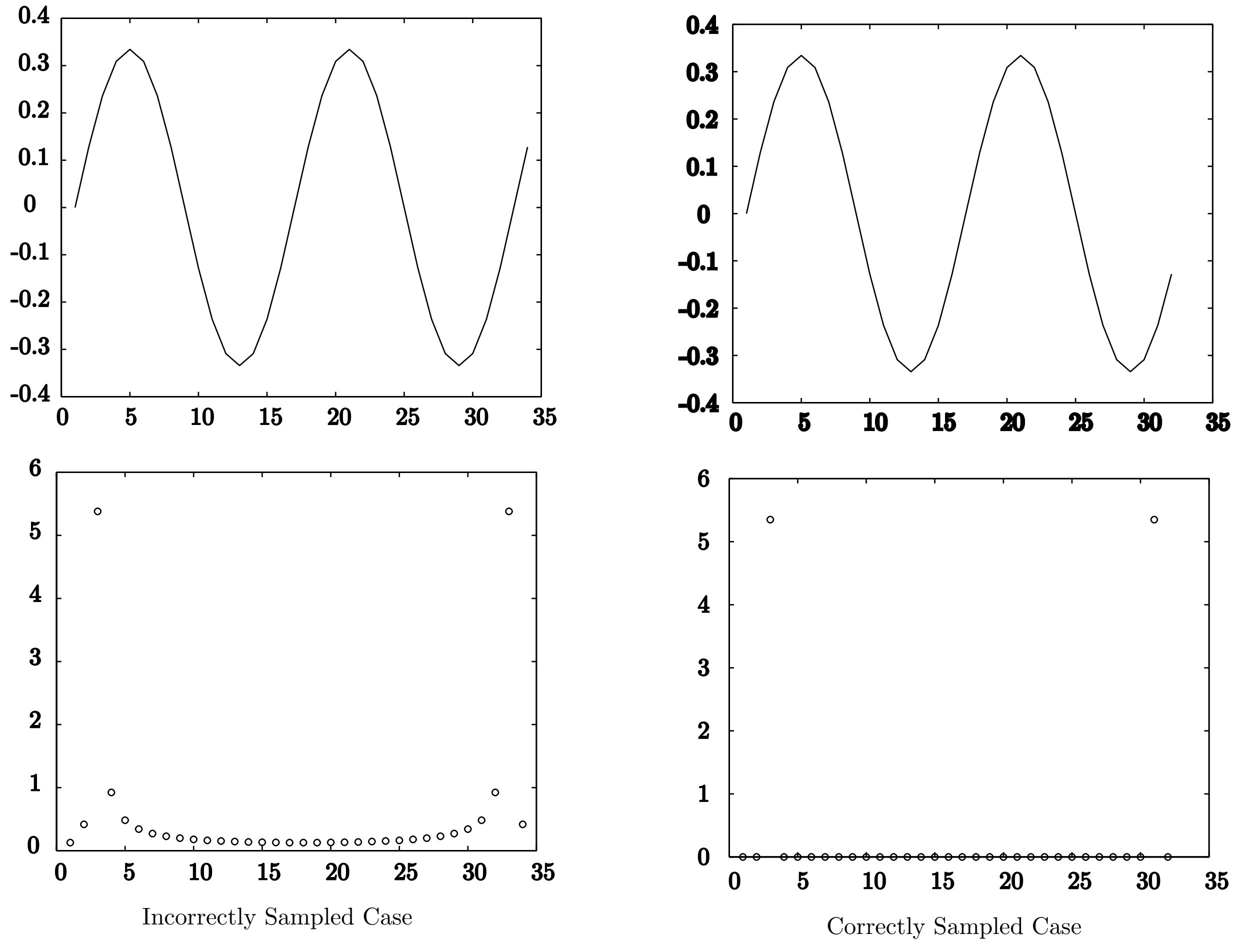


Figure 6. Fourier Transforms of $\sin(2\pi f\ t)$

The only difference between the incorrectly sampled vs. correctly cases is the number of samples used, 32 vs. 34. The nonzero magnitude in the incorrectly samped case is known as *leakage phenomenon* in FFT-based data processing. This and related *filtering techniques* become an integral part of system identification activities, which we plan to cover in the subsequent sections.

§17.5 IMPULSE RESPONSES WHEN LOADINGS ARE ARBITRARY

In the preceding sections we have obtained the *reference* impulse response functions both in frequency and time domains when the loadings are the exact unit impulse function. This is indeed a luxuary that is difficult to realize in practice. In a laboratory testing environment (we will address the in-situ loads later), one creates a series of ramdom bursts that are rich in frequency contents. To simulate such loadings, fortuntely using MATLAB©, one can generate them by invoking `randn` command:

$$\begin{aligned} \mathbf{f}_1 &= \texttt{5 * randn(1,512)} \\ \mathbf{f}_2 &= \texttt{10 * randn(1,512)} \\ \mathbf{f}_3 &= \texttt{100* randn(1,512)} \end{aligned} \tag{17.27}$$

It should be noted that the three random number sets in the above command would in principle be different, since at each time Matlab generates a new set of random numbers!

Applying the above three forcing functions, the transient response of the three degrees of freedom oscillator can be obtained as shown in Figs. 7 and 8.

The frequency domain system description of the 3dof oscillator for this forcing function case can be obtained by using (17.12):

$$\begin{aligned} &\mathbf{Q}(\omega) = \int_{-\infty}^{\infty} \mathbf{q}(t) e^{-j\omega t}\, dt = \int_{-\infty}^{\infty} (-\boldsymbol{\omega}^2 \mathbf{M} \, + \mathbf{K})^{-1} \mathbf{f}(t) e^{-j\omega t}\, dt \\ &\qquad\Downarrow \\ &\mathbf{Q}(\omega) = \mathbf{H}(\omega) * \mathbf{f}(\omega) \end{aligned} \tag{17.28}$$

In the above expressions, $*$ is the frequency-domain counterpart of the time convolution integral. The impulse response function $\mathbf{H}(\omega)$ can thus be obtained, *only symbolically*, by

$$\mathbf{H}(\omega) = [\mathbf{Q}(\omega) * \mathbf{f}^T(\bar{\omega})]\, [\mathbf{f}(\omega) * \mathbf{f}^T(\bar{\omega})]^{-1} \qquad \bar{\omega} = \text{conjugate of } \omega \tag{17.29}$$

This task, especially with experimental data, remains a challenge. A good portion of this course will be devoted to the extraction of accurate frequency response functions from multi-input and multi-output (MIMO) systems.

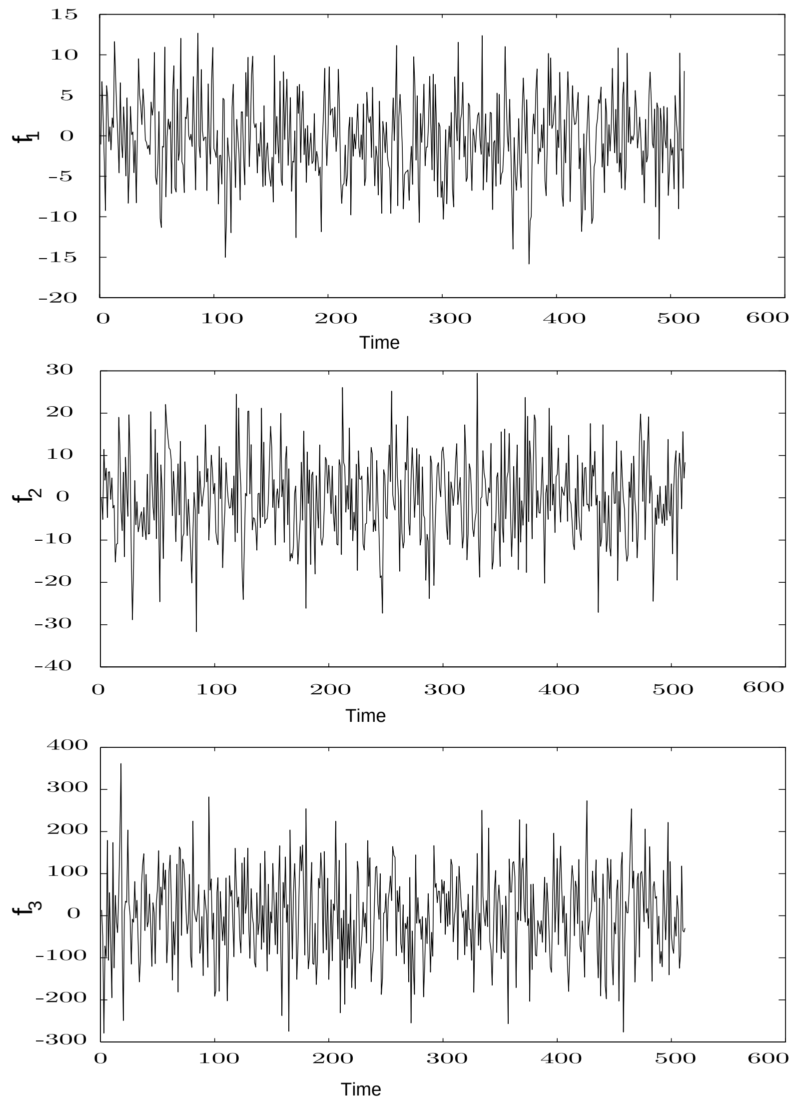


Figure 7. Randomly Generated Forcing Functions

§17.6 A CLASSICAL IDENTIFICATION METHOD

Suppose that we have obtained the impulse response functions either in frequency domain (Fig. 4) or in time domain (Fig. 5) from measurement data. It should be noted that the transient response histories under random forcing functions shown in Fig. 8 are typical of measured data obtained by vibration tests. We now want to obtain the modes $\boldsymbol{\omega}$ and the mode shapes $\boldsymbol{\phi}$ from the impulse response function $\mathbf{H}(\omega)$ (17.11) or $\mathbf{h}(t)$ (17.22).

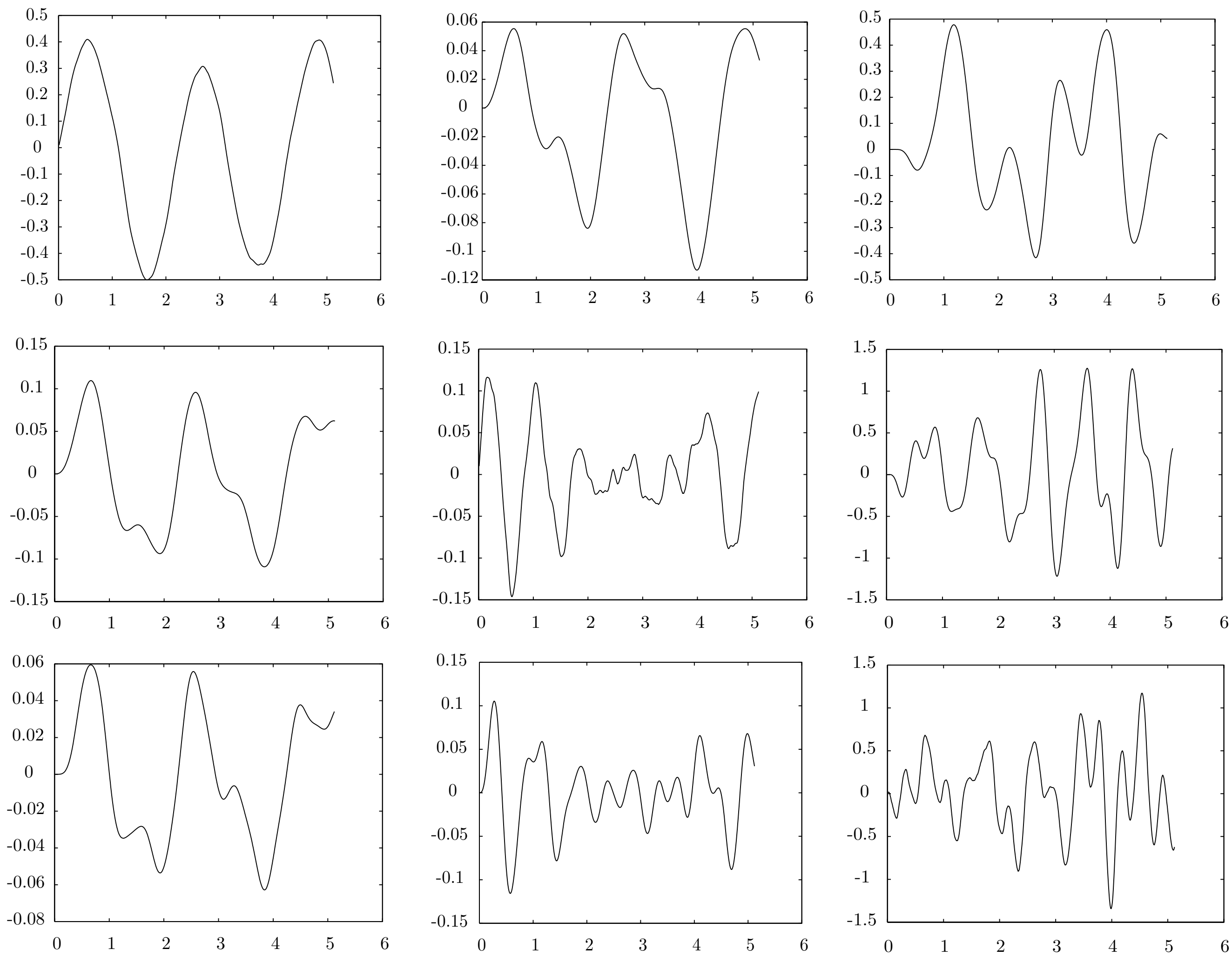


Figure 8.Transient Rsponse of 3-DOF System under Random Forcing Functions

§1.6.1 Frequency Method

The frequency response function (17.10) can be reexpressed by using (17.18) as

$$\begin{aligned}\mathbf{H}(\omega) &= (-\boldsymbol{\omega}^2\, \mathbf{M} + \mathbf{K})^{-1} \\ &= (-\boldsymbol{\omega}^2\, \boldsymbol{\phi}\, \boldsymbol{\phi}^T + \boldsymbol{\phi}\, \boldsymbol{\Omega}^2\, \boldsymbol{\phi}^T)^{-1} \\ &= \boldsymbol{\phi}\, (\, -\boldsymbol{\omega}^2 \,+\, \boldsymbol{\Omega}^2\,)^{-1} \boldsymbol{\phi}^T\end{aligned} \tag{17.30}$$

Therefore, the component-by-component frequency response function $\mathbf{H}(\omega)_{ij}$ is given by

$$\mathbf{H}_{ij}(\omega) = \sum_{k=1}^{N} \frac{\phi_{ik}\,\phi_{jk}}{(\Omega_k^2 - \omega^2)} = \sum_{k=1}^{N} \frac{{}_kA_{ij}}{(\Omega_k^2 - \omega^2)} \tag{17.31}$$

The denominators in the above equation play an important role in the identification of vibrating structures. In this course it will be called a *modal constant* or *modal residual constant,* given by

$$\text{Modal (Residual) Constant:} \qquad {}_kA_{ij} = \phi_{ik}\,\phi_{jk} \tag{17.32}$$

Observe that, for multi-degrees of freedom systems, each modal constant magnitude in the modal series expansion of $\mathbf{H}(\omega)_{ij}$ dictates the influence of that mode. For example, one may truncate certain modal contributions if their modal constants are sufficiently small. As an example, let us compute the three modal constants of $\mathbf{H}_{11}(\omega)$ from the 3-DOF example problem given by

$$H_{11}(\omega) = \sum_{k=1}^{N} \frac{\phi_{1k}\,\phi_{1k}}{(\Omega_k^2 - \omega^2)} = \sum_{k=1}^{N} \frac{{}_kA_{11}}{(\Omega_k^2 - \omega^2)} \tag{17.33}$$

where ${}_kA_{ij}$ can be obtained from (17.3) as

$$\begin{aligned} {}_1A_{11} &= \phi_{11}^2 = 0.974189634^2 = 9.490454429\text{E-01} \\ {}_2A_{11} &= \phi_{21}^2 = 0.224712761^2 = 5.049582495\text{E-02} \\ {}_3A_{11} &= \phi_{31}^2 = 0.021417986^2 = 4.587301242\text{E-04} \end{aligned} \tag{17.34}$$

For convenience, Fig. 4a is reproduced below for discussion.

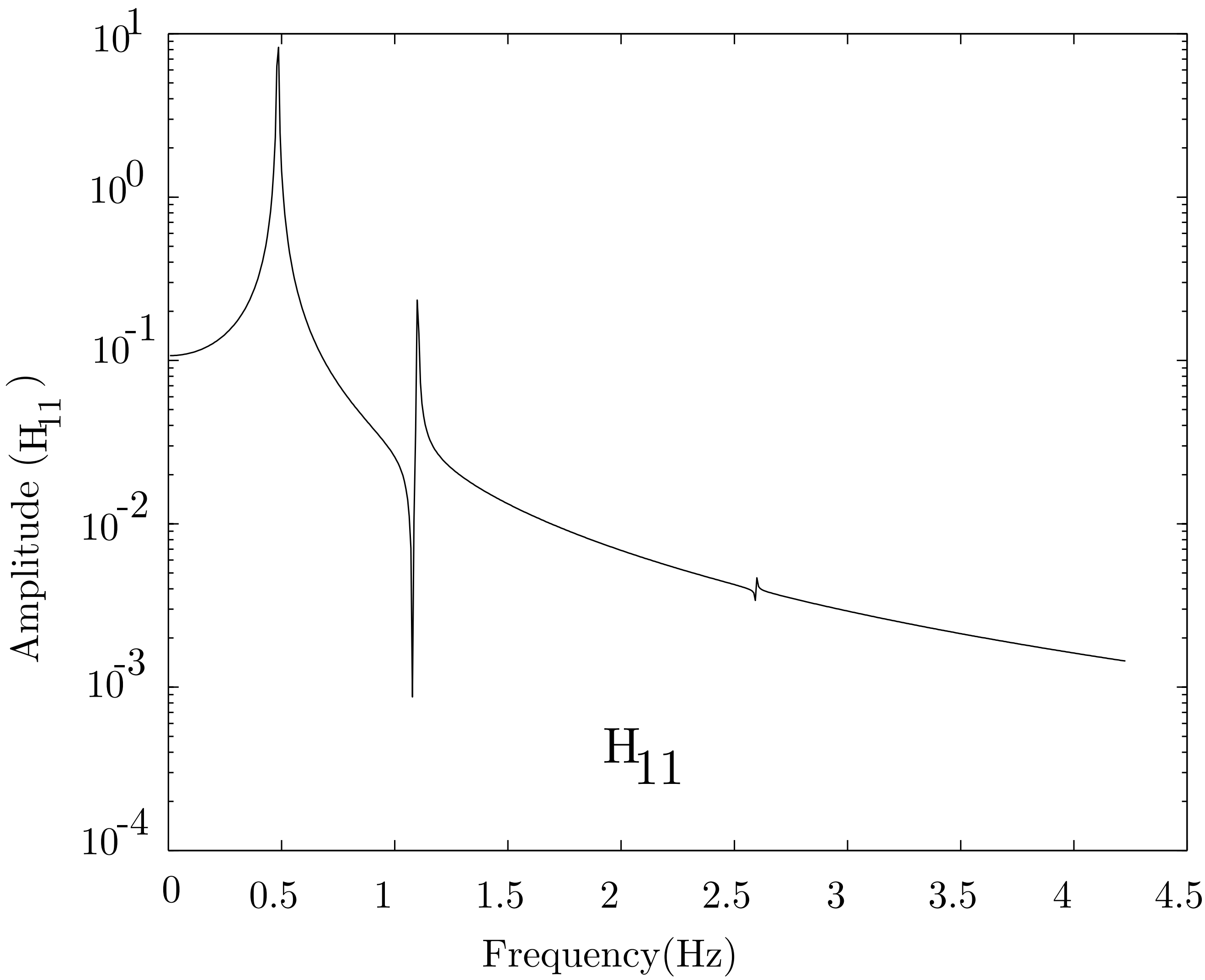


Figure 9. Impulse Frequency Function $\mathbf{H}(\omega)_{11}$

Notice that the fact that ${}_3A_{11} = 4.587301242\text{E-}04$ is small compared with the other two modal constants. This is reflected as a small blip in Fig. 9 at the frequency $\Omega_3 = 16.2719/2\pi = 2.5898\ Hz$.

Now suppose that we have obtained the impulse frequency response curves $\mathbf{H}(\omega)_{ij}$ from measurements. The first step in constructing the experimentally determined model is to obtain the modal constants ${}_kA_{11}$. To determine the three modal constants ${}_1A_{11}$, ${}_2A_{11}$ and ${}_3A_{11}$, one constructs the following matrix relation for all the discrete points $\omega_{min} \le \omega \le \omega_{max}$:

$$\begin{bmatrix} \mathbf{H}_{11}(\omega_{min}) \\ \mathbf{H}_{11}(\omega_2) \\ \mathbf{H}_{11}(\omega_3) \\ . \\ . \\ \mathbf{H}_{11}(\omega_{max}) \end{bmatrix} = \begin{bmatrix} (\Omega_1^2 - \omega_{min}^2)^{-1} & (\Omega_2^2 - \omega_{min}^2)^{-1} & (\Omega_3^2 - \omega_{min}^2)^{-1} \\ (\Omega_1^2 - \omega_2^2)^{-1} & (\Omega_2^2 - \omega_2^2)^{-1} & (\Omega_3^2 - \omega_2^2)^{-1} \\ (\Omega_1^2 - \omega_3^2)^{-1} & (\Omega_2^2 - \omega_3^2)^{-1} & (\Omega_3^2 - \omega_3^2)^{-1} \\ . & & \\ . & & \\ (\Omega_1^2 - \omega_{max}^2)^{-1} & (\Omega_2^2 - \omega_{max}^2)^{-1} & (\Omega_3^2 - \omega_{max}^2)^{-1} \end{bmatrix} \begin{bmatrix} {}_1A_{11} \\ {}_2A_{11} \\ {}_3A_{11} \end{bmatrix}$$

$$\Downarrow$$

$$\mathbf{H}_{11}^{v}(n \times 1) = \mathbf{R}(n \times 3)\ \mathbf{A}(3 \times 1) \tag{17.35}$$

where $\mathbf{H}_{11}^{v}$ denotes that $\{H_{11}(\omega), \quad \omega_{min} \le \omega \le \omega_{max}\}$ is arranged as a vector.

A least squares solution of the above overdetermined equation would give the sought-after modal constants for $\mathbf{H}(\omega)_{11}$

$$\begin{bmatrix} {}_1A_{11} \\ {}_2A_{11} \\ {}_3A_{11} \end{bmatrix} = [\mathbf{R}^T\mathbf{R}]^{-1}\ [\mathbf{R}^T\mathbf{H}_{11}^{v}] \tag{17.36}$$

The modal constants for the remaining eight frequency response components can be similarly determined. Finally, the mode shape ϕ which has nine unknowns can be solved by a nonlinear programming method once the eighteen modal constants are determined (Remember we have six distinct frequency response functions H_{ij}, namely, H_{11}, H_{12}, H_{13}, H_{22}, H_{23} and H_{33}. And each has three modal constants).

§1.6.2 A Classical Time Domain Method

There exist several classical time domain methods for determining the modes and mode shapes. Among a plethora of classical methods, we will study Ibrahim's method as it can be viewed as a precursor to modern *realization algorithm* based on Kalman's minimum realization procedure.

The transient response for a first-order system $\mathbf{x}(t)$, see (17.14), can be expressed from (17.15)

$$\mathbf{x}(t) = \mathbf{e}^{\mathcal{A}t}\mathbf{x}(0), \qquad \mathcal{A} = \begin{bmatrix} \mathbf{0} & -\mathbf{K} \\ \mathbf{M}^{-1} & \mathbf{0} \end{bmatrix} \tag{17.37}$$

Since we have

$$\mathcal{A} = \boldsymbol{\Psi}\,\boldsymbol{\lambda}\,\boldsymbol{\Psi}^{-1}, \qquad \Rightarrow \qquad \mathbf{e}^{\mathcal{A}} = \boldsymbol{\Psi}\,\mathbf{e}^{\boldsymbol{\lambda}}\,\boldsymbol{\Psi}^{-1}$$

$$\mathbf{e}^{\boldsymbol{\lambda}} = \begin{bmatrix} \mathbf{e}^{\lambda_1} & . & . & . & . \\ . & \mathbf{e}^{\lambda_2} & . & . & . \\ . & . & \mathbf{e}^{\lambda_3} & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & \mathbf{e}^{\lambda_n} \end{bmatrix} \tag{17.38}$$

the coupled transient response $\mathbf{x}(t)$ can be expressed in terms of the first-order eigenvalue $\boldsymbol{\lambda}$ and the associated eigenvector $\boldsymbol{\Psi}$:

$$\mathbf{x}(t) = \boldsymbol{\Psi}\,\mathbf{e}^{\boldsymbol{\lambda}t}\,\boldsymbol{\Psi}^{-1}\mathbf{x}(0) = \boldsymbol{\Psi}\,\mathbf{e}^{\boldsymbol{\lambda}t}\,\mathbf{z}(0), \quad \mathbf{x}(t) = \boldsymbol{\Psi}\,\mathbf{z}(t) \tag{17.39}$$

which for the example 3-DOF system becomes

$$\underset{(6\times 1)}{\mathbf{x}(t)} = \underset{(6\times 6)}{\boldsymbol{\Psi}} \quad \underset{(6\times 6)}{\mathbf{e}^{\boldsymbol{\lambda}t}} \quad \underset{(6\times 1)}{\mathbf{z}(0)} \tag{17.40}$$

Therefore, for n-discrete response points one has the following relation:

$$\begin{gathered} [\,\mathbf{x}(t_1) \quad \mathbf{x}(t_2) \quad \ldots\,\mathbf{x}(t_n)\,] = \boldsymbol{\Psi}\,\big[\,[\mathbf{e}^{\boldsymbol{\lambda}t_1}] \quad [\mathbf{e}^{\boldsymbol{\lambda}t_2}] \quad \ldots\,[\mathbf{e}^{\boldsymbol{\lambda}t_n}]\,\big]\,\mathbf{z}(0) \\ \Downarrow \\ \mathbf{X}(t_1) = \boldsymbol{\Psi}\,\boldsymbol{\Lambda}(t_1)\,\mathbf{z}(0) \end{gathered} \tag{17.41}$$

Now if we shift the starting sampling event from $t = t_1$ to $t_2 = t_1 + \Delta t$, we have a time-shifted equation

$$\begin{gathered} [\,\mathbf{x}(t_2) \quad \mathbf{x}(t_3) \quad \ldots\,\mathbf{x}(t_{n+1})\,] = \boldsymbol{\Psi}\,\big[\,[\mathbf{e}^{\boldsymbol{\lambda}t_2}] \quad [\mathbf{e}^{\boldsymbol{\lambda}t_3}] \quad \ldots\,[\mathbf{e}^{\boldsymbol{\lambda}t_{n+1}}]\,\big]\,\mathbf{z}(0) \\ \Downarrow \\ \mathbf{X}(t_2) = \boldsymbol{\Psi}\,\boldsymbol{\Lambda}(t_2)\,\mathbf{z}(0) \end{gathered} \tag{17.42}$$

Now observe that $[\mathbf{e}^{\boldsymbol{\lambda} t_{i+1}}]$ can be decomposed as

$$[\mathbf{e}^{\boldsymbol{\lambda} t_{i+1}}] = \mathbf{E}(\Delta t)\,[\mathbf{e}^{\boldsymbol{\lambda} t_i}]$$

$$\mathbf{E}(\Delta t) = \begin{bmatrix} \mathbf{e}^{\lambda_1 \Delta t} & . & . & . & . & . \\ . & \mathbf{e}^{\lambda_2 \Delta t} & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & \mathbf{e}^{\lambda_6 \Delta t} \end{bmatrix} \tag{17.43}$$

Using this decomposition and (17.41), $\mathbf{X}(t_2)$ in (17.42) can be written as

$$\begin{aligned} \mathbf{X}(t_2) &= \boldsymbol{\Psi}\,\boldsymbol{\Lambda}(t_2)\,\mathbf{z}(0) = \boldsymbol{\Psi}\,\mathbf{E}(\Delta t)\,\boldsymbol{\Lambda}(t_1)\,\mathbf{z}(0) = \boldsymbol{\Psi}\,\mathbf{E}(\Delta t)\,\boldsymbol{\Psi}^{-1}\,\mathbf{X}(t_1) \\ &\Downarrow \\ \mathbf{X}(t_2) &= \mathbf{S}(\Delta t)\,\mathbf{X}(t_1) \\ \mathbf{S}(\Delta t) &= \boldsymbol{\Psi}\,\mathbf{E}(\Delta t)\,\boldsymbol{\Psi}^{-1} \end{aligned} \tag{17.44}$$

Finally, one obtains

$$\mathbf{S}(\Delta t) = [\mathbf{X}(t_2)\,\mathbf{X}^T(t_1)]\,[\mathbf{X}(t_1)\,\mathbf{X}^T(t_1)\,]^{-1} \tag{17.45}$$

Once $\mathbf{S}(\Delta t)$ is determined, the system eigenvalues and eigenvectors can be obtained from the following eigenproblem:

$$\mathbf{S}(\Delta t)\boldsymbol{\Psi} = \boldsymbol{\Psi}\,\mathbf{E}(\Delta t) \tag{17.46}$$

It should be noted that the resulting eigenvectors are not scaled in general, especially with respect to the system mass matrix. This and other issues will be discussed in the following chapters.

§17.7 DISCUSSIONS

As stated in Section §17.2, *structural analysis* is first to construct the structural models, viz., the mass, the damping, and stiffness operators. Then by using known forcing functions or best possible forcing function models, it is to determine the response of the structure. *Structural system identification* on the other hand is to determine the structural model or model parameters based on the measured forcing functions and the measured structural response data. More precisely, let us consider a general system model given by (we will derive them in the next chapter)

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathcal{A}\,\mathbf{x}(t) + \mathcal{B}\,\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathcal{C}\,\mathbf{x}(t) + \mathcal{D}\,\mathbf{u}(t) \end{aligned} \tag{17.47}$$

Structural analysis is to obtain $\mathbf{x}$ for a given $\mathbf{u}$ *knowing* that $\mathcal{A}$ and $\mathcal{B}$ are known by modeling work, usually by the finite element method or the boundary element method. Structural system identification is to obtain $\mathcal{A}$ along with the remaining three opeartors $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$, when $\mathbf{u}$ and the sensor output $\mathbf{y}$ are available. In other words, structural analysis is to obtain vectorial quantities from known matrix quantities, whereas structural system identification is to obtain matrix quantities from measured vectorial quantities.

Therefore, the structural response can be obtained uniquely once the structural model (structural mass, stiffness and damping matrices) and the forcing function are given. This is not the case in general in determining the structural model from measured forcing function and measured response data. This is because the information (usually frequency contents) contained in the measured response can vary widely, depending on the sensor types, sensor locations, and forcing function characteristics. For example, if the forcing function consists of two tuned harmonic frequencies, the measured response would capture at most two modes by tuning the forcing function frequencies to match two of the system modes. This is why one prefers to utilize forcing functions that contain rich enough frequency contents. This and other related issues will be discussed throughout the course.